Devaney chaos in non-autonomous discrete systems

Hao Zhu, Yuming Shi†, Hua Shao

Department of Mathematics, Shandong University
Jinan, Shandong 250100, P. R. China

Abstract. This paper is concerned with Devaney chaos in non-autonomous discrete systems. It is shown that in its definition, the two former conditions, i.e., transitivity and density of periodic points, in a set imply the last one, i.e., sensitivity, in the case that the set is unbounded, while a similar result holds under two additional conditions in the other case that the set is bounded. Furthermore, some chaotic behavior is studied for a class of non-autonomous systems, each of which is governed by a convergent sequence of continuous maps.

Keywords: non-autonomous discrete system; Devaney chaos; transitivity; periodic point; sensitivity.

2010 *Mathematics Subject Classification*: 37B55; 37D45; 54H20.

1. Introduction

In this paper, we consider the following non-autonomous discrete system:

$$x_{n+1} = f_n(x_n), \quad n \ge 0, \tag{1.1}$$

where $f_n : D_n \to D_{n+1}$ is a map and D_n is a subset of a metric space (X, d) for each $n \ge 0$. When $f_n = f$ and $D_n = D$ for each $n \ge 0$, system (1.1) becomes the following autonomous discrete system:

$$x_{n+1} = f(x_n), \quad n \ge 0. \tag{1.2}$$

Chaos, which is one of the central topics of research on nonlinear science, reflects the complexity of dynamical systems. Chaos theory of autonomous systems has been extensively studied [7, 17, 25] and there appears several definitions of chaos in autonomous systems [5, 7, 15, 18, 25]. In particular, Devaney in 1989 gave the following definition of chaos for system (1.2):

Definition 1.1 [7]. Let X be a metric space. System (1.2) is said to be chaotic on X if

† The corresponding author

Email addresses: haozhu@mail.sdu.edu.cn(H. Zhu), ymshi@sdu.edu.cn(Y. Shi), huashaosdu@163.com (H. Shao).

This research was supported by the NNSF of China (Grant 11571202).

(1) system (1.2) is topologically transitive in X;

(2) the periodic points of system (1.2) is dense in X;

(3) system (1.2) has sensitive dependence on initial conditions in X.

In 1992, Banks and his coauthors proved that (1) and (2) imply (3) in the above definition if f is continuous in X [3]. This shows that condition (3) in this definition is redundant in this case. For further results, see [10, 12].

Non-autonomous systems occur more often than autonomous ones since many physical, biological, and economical problems are often described by non-autonomous systems. Thus, the study on complexity of system (1.1) has been of increasing interest by many researchers in recent years [2, 4, 6, 8, 11, 14, 16, 19–24, 26]. We briefly recall some existing results related to Devaney chaos for system (1.1). In 2006, Tian and Chen studied system (1.1) in the special case that $D_n = X$ for each $n \geq 0$ and extended the concept of Devaney chaos to system (1.1) [24]. In 2009, Shi and Chen removed the assumption that $D_n = X$ for each $n \geq 0$ and thus generalized the concept of Devaney chaos for system (1.1) [21] (See also Definition 2.4). There appears a natural question: whether the two former conditions–transitivity and density of periodic points imply the last one–sensitivity for system (1.1). This is the main topic studied in the present paper.

Chaotic behavior of system (1.1) was studied in the special case that $\{f_n\}_{n=0}^{\infty}$ converges to a map f by some scholars (cf., [6, 8, 26] and references cited therein). Cánovas investigated the relationships between Li-Yorke chaos of system (1.1) and that of system (1.2) for interval maps [6]. Dvořáková showed that system (1.2) may not be distributionally chaotic even if system (1.1) has a distributionally scrambled set with full Lebesgue measure [8]. In the present paper, we shall study Devaney chaos of system (1.1) in this special case.

The rest of this paper is organized as follows. In Section 2, some basic concepts and lemmas are given. In particular, properties of the three conditions of Devaney chaos for system (1.1) are investigated, some of which are shown by different methods from those in the autonomous case. In Section 3, it is shown that the two former conditions in a set V imply the last one of Devaney chaos for system (1.1) in the case that V is unbounded, while a similar result holds under two additional conditions in the other case that V is bounded. In the special case that the sequence of continuous maps $\{f_n\}_{n=0}^{\infty}$ converges to some continuous map f, some chaotic behavior of system (1.1) is investigated by using that of system (1.2) in Section 4.

2. Preliminaries

In this section, some basic concepts and lemmas are given. Especially, similarities and differences between properties of the three conditions of Devaney chaos for system (1.1) and those for system (1.2) are presented.

For convenience, denote $f_{0,\infty} := \{f_n\}_{n=0}^{\infty}$, $f_n^i := f_{n+i-1} \circ \cdots \circ f_n$, and $f_n^0 := id$. For a fixed $x_0 \in D_0$, set $O(x_0) := \{f_0^i(x_0) : i \geq 0\}$. Let $x \in X$ and $\varepsilon > 0$. By $B_\varepsilon(x)$ and $\overline{B}_\varepsilon(x)$ denote the open and closed balls of radius ε, centered at x, respectively.

Definition 2.1 [21, Definition 2.1]. A point $p \in D_0$ is called k-periodic for system (1.1) if $f_0^{n+k}(p) = f_0^n(p), n \geq 0$. Further, k is called the prime period of p if $f_0^i(p) \neq p$, $1 \leq i \leq k-1$. In the special case of $k = 1$, i.e., $f_n(p) = p$ for all $n \geq 0$, p is called a fixed point of system (1.1). The set of all the periodic points is denoted by $P(f_{0,\infty})$.

Remark 2.1. It is easy to see that two periodic orbits of system (1.2) either do not intersect or agree completely. However, it is not true for system (1.1) in general. See the following example:

Example 2.1. Let $X = \{a, b, c, d\}$ be equipped with discrete metric. Set $D_{4n} = \{a, b\}$, $D_{4n+1} = \{b, c\}$, $D_{4n+2} = \{a, d\}$, $D_{4n+3} = \{a, b\}$, $f_{2n}(a) = b$, $f_{2n+1}(b) = a$, $f_{4n}(b) = c$, $f_{4n+1}(c) = d$, $f_{4n+2}(d) = a$, and $f_{4n+3}(a) = b$ for each $n \geq 0$. Then a is a 2-periodic point and $O(a) = \{a, b\}$; b is a 4-periodic point and $O(b) = \{b, c, d, a\}$. It is evident that $O(a) \cap O(b) = \{a, b\}$ and $O(a) \neq O(b)$.

Let V be a subset of D_0. For any two nonempty relatively open subsets U_0 and V_0 of V, denote $N(U_0, V_0) := \{n \in \mathbb{N}_+ : f_0^n(U_0) \cap V_0 \neq \emptyset\}$.

Definition 2.2 [21, Definition 2.2]. Let V be a nonempty subset of D_0. System (1.1) is said to be topologically transitive in V if $N(U_0, V_0) \neq \emptyset$ for any two nonempty relatively open subsets U_0 and V_0 of V.

Remark 2.2. For system (1.2), it is easy to show that if it is topologically transitive in a finite set V, then V consists of a periodic orbit. However, it is not true for system (1.1) in general.

Example 2.2. Let $X = \{a, b\}$ be equipped with discrete metric. Set $V = D_n = X$ for each $n \geq 0$, $f_0(a) = f_1(a) = b$, $f_0(b) = f_1(b) = a$, $f_m(a) = a$, and $f_m(b) = b$ for each $m \geq 2$. Then system (1.1) is topologically transitive in V. However, there are no periodic points in V.

It is known that if system (1.2) is topologically transitive in an infinite set V, then V contains no isolated points [13]. For system (1.1), a similar result is not true in general.

Example 2.3. Let $X = [0, 1] \cup \{2\}$ be equipped with the induced topology from the Euclidean space $\mathbb{R}$ and $\{a_n\}_{n=0}^{\infty}$ be the sequence of rational numbers in $[0, 1]$. Set $V =$

$D_n = X$, $f_{2n}(x) = a_n$, and $f_{2n+1}(x) = 2$ for each $n \geq 0$ and $x \in V$. Though system (1.1) is topologically transitive in the infinite set V, V contains an isolated point 2.

However, we have the following result, which will be useful in the sequent sections:

Lemma 2.1. *Let V be an infinite subset of D_0. If system* (1.1) *is topologically transitive in V and $P(f_{0,\infty}) \cap V$ is dense in V, then there are no isolated points in V.*

Proof. Suppose that there exists an isolated point $x_0 \in V$. Then there exists $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x_0) \cap V = \{x_0\}$. Since $P(f_{0,\infty}) \cap V$ is dense in V, x_0 is a periodic point. So one can choose $y_0 \in V \backslash O(x_0)$ due to the fact that V is infinite. Let $\delta_0 := d(y_0, O(x_0)) > 0$. Then $f_0^n(B_{\varepsilon_0}(x_0) \cap V) \cap (B_{\delta_0/3}(y_0) \cap V) = \emptyset$ for each $n \geq 0$, which contradicts the assumption that system (1.1) is topologically transitive in V. This completes the proof.

Definition 2.3 [21, Definition 2.3]. Let V be a nonempty subset of D_0. System (1.1) is said to have sensitive dependence on initial conditions (briefly, system (1.1) is sensitive) in V if there exists a constant $\delta_0 > 0$ such that for any $x_0 \in V$ and any neighborhood U of x_0, there exist $y_0 \in V \cap U$ and a positive integer n such that $d(f_0^n(x_0), f_0^n(y_0)) > \delta_0$. The constant δ_0 is called a sensitivity constant of system (1.1) in V.

Remark 2.3. It is easy to see that if there exists an isolated point in V, then system (1.1) is not sensitive in V.

Definition 2.4 [21, Definition 2.5]. Let V be a nonempty subset of D_0. System (1.1) is said to be chaotic in the sense of Devaney on V if

(1) system (1.1) is topologically transitive in V;

(2) $P(f_{0,\infty}) \cap V$ is dense in V;

(3) system (1.1) has sensitive dependence on initial conditions in V.

The next lemma shows that (1) and (2) imply (3) of Devaney chaos for a continuous map.

Lemma 2.2 [3]. Let X be a metric space and infinite, and $f : X \to X$ be a continuous map. If f is topologically transitive and has dense periodic points in X, then f is sensitive in X.

Definition 2.5 [21, Definition 2.4]. If V is a nonempty subset of $\bigcap_{n=0}^{\infty} D_n \neq \emptyset$ satisfying $f_0^n(V) \subset V$ for all $n \geq 0$, then V is called a totally invariant set of system (1.1).

Definition 2.6 [22, Definition 2.6]. Assume that D_n and E_n are two subsets of a metric space (X, d), and $h_n : D_n \to E_n$ is a map for each $n \geq 0$. The sequence of maps $\{h_n\}_{n=0}^{\infty}$ is said to be equi-continuous in $\{D_n\}_{n=0}^{\infty}$ if for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that $d(h_n(x), h_n(y)) < \varepsilon$ for all $x, y \in D_n$ with $d(x, y) < \delta$ and for all $n \geq 0$.

Definition 2.7. An increasing sequence of nonnegative integers $\{n_k\}_{k=1}^{\infty}$ is said to be syndetic if there exists an integer $l \geq 1$ such that $n_{k+1} - n_k \leq l$ for all $k \geq 1$.

Now we introduce the concept of topological ergodicity for system (1.1).

Definition 2.8. Let V be a nonempty subset of D_0. System (1.1) is said to be topologically ergodic in V if $N(U_0, V_0)$ is syndetic for any two nonempty relatively open subsets U_0 and V_0 of V.

It was shown that system (1.2) is topologically ergodic if it satisfies (1) and (2) in Definition 1.1 (See [1] or [9]). The next result shows that a similar result is true in the non-autonomous case.

Theorem 2.1. *Let V be a totally invariant set of system* (1.1). *Assume that f_n is continuous in V for each $n \geq 0$. If system* (1.1) *is topologically transitive in V and $P(f_{0,\infty}) \cap V$ is dense in V, then for any two nonempty relatively open subsets U_0 and V_0 of V, there exists $p_0 \in U_0 \cap P(f_{0,\infty})$ such that $O(p_0) \cap V_0 \neq \emptyset$. Consequently, system* (1.1) *is topologically ergodic in V.*

Proof. Let U_0 and V_0 be two nonempty relatively open subsets of V. Since system (1.1) is topologically transitive in V, there exist $x_0 \in U_0$ and $n_0 > 0$ such that $f_0^{n_0}(x_0) \in V_0$. Fix $\varepsilon > 0$ satisfying that $B_\varepsilon(f_0^{n_0}(x_0)) \cap V \subset V_0$. By the continuity of $f_0^{n_0}$ in V and the assumption that V is a totally invariant set of system (1.1), there exists $\delta > 0$ such that $B_\delta(x_0) \cap V \subset U_0$ and $f_0^{n_0}(B_\delta(x_0) \cap V) \subset B_\varepsilon(f_0^{n_0}(x_0)) \cap V \subset V_0$. Since $P(f_{0,\infty}) \cap V$ is dense in V, there exists $p_0 \in B_\delta(x_0) \cap V \cap P(f_{0,\infty}) \subset U_0$ with period m_0. Then $f_0^{n_0+km_0}(p_0) = f_0^{n_0}(p_0) \in V_0$ for each $k \geq 0$. Thus, $O(p_0) \cap V_0 \neq \emptyset$ and $\{n_0 + km_0\}_{k=0}^{\infty} \subset N(U_0, V_0)$, which implies that $N(U_0, V_0)$ is syndetic. Hence, system (1.1) is topologically ergodic in V. This completes the proof.

3. Transitivity and density of periodic points imply sensitivity

In this section, we shall show that transitivity and density of periodic points imply sensitivity in V for system (1.1) under certain conditions. The discussions are divided into the two cases: V is unbounded and bounded.

In the case that V is unbounded, we get the following result:

Theorem 3.1. *Let V be an unbounded subset of D_0. If system* (1.1) *is topologically transitive in V and $P(f_{0,\infty}) \cap V$ is dense in V, then it is sensitive in V and the sensitivity constant can be an arbitrarily positive number.*

Proof. Let $\delta > 0$ and $a \geq 2$ be two arbitrarily given number, x be an arbitrary point in V, and $U(x)$ be any neighborhood of x in X. Then we show that there exist $y_0 \in U(x) \cap V$

and an integer $m_0 > 0$ such that

$$d(f_0^{m_0}(x), f_0^{m_0}(y_0)) > (a-1)\delta/2.$$

Since system (1.1) is topologically transitive in V and $P(f_{0,\infty}) \cap V$ is dense in V, by Lemma 2.1 there are no isolated points in V and thus there exists a periodic point $q \in U(x) \cap V$ such that $q \neq x$. Set

$$L_{x,q} := \max_{z \in O(q)} d(x, z) > 0. \tag{3.1}$$

Then the discussions are divided into two cases.

Case 1. $L_{x,q} \geq \delta$.

By (3.1), $O(q) \subset \overline{B}_{L_{x,q}}(x)$. Since V is unbounded, $\big(X \backslash \overline{B}_{aL_{x,q}}(x)\big) \cap V$ is a nonempty open subset of V. By the transitivity of system (1.1) in V, there exist $y_1 \in U(x) \cap V$ and an integer $m_1 > 0$ such that $f_0^{m_1}(y_1) \in \big(X \backslash \overline{B}_{aL_{x,q}}(x)\big) \cap V$. Then

$$d(f_0^{m_1}(q), f_0^{m_1}(y_1)) \geq d(x, f_0^{m_1}(y_1)) - d(x, f_0^{m_1}(q)) > aL_{x,q} - L_{x,q} \geq (a-1)\delta.$$

Thus, one gets that

$$\text{either } \; d(f_0^{m_1}(x), f_0^{m_1}(y_1)) > (a-1)\delta/2 \; \text{ or } \; d(f_0^{m_1}(x), f_0^{m_1}(q)) > (a-1)\delta/2. \tag{3.2}$$

Case 2. $L_{x,q} < \delta$.

Since V is unbounded, $\big(X \backslash \overline{B}_{a\delta}(x)\big) \cap V$ is a nonempty open subset of V. As system (1.1) is topologically transitive in V, there exist $y_2 \in U(x) \cap V$ and an integer $m_2 > 0$ such that $f_0^{m_2}(y_2) \in \big(X \backslash \overline{B}_{a\delta}(x)\big) \cap V$. This, together with the fact that $f_0^{m_2}(q) \in \overline{B}_{L_{x,q}}(x) \subset \overline{B}_{\delta}(x)$, implies that

$$d(f_0^{m_2}(q), f_0^{m_2}(y_2)) \geq d(x, f_0^{m_2}(y_2)) - d(x, f_0^{m_2}(q)) > a\delta - \delta = (a-1)\delta.$$

Thus,

$$\text{either } \; d(f_0^{m_2}(x), f_0^{m_2}(y_2)) > (a-1)\delta/2 \; \text{ or } \; d(f_0^{m_2}(x), f_0^{m_2}(q)) > (a-1)\delta/2. \tag{3.3}$$

It follows from (3.2) and (3.3) that system (1.1) is sensitive in V with sensitivity constant $(a-1)\delta/2$. Since $a \geq 2$ and $\delta > 0$ are arbitrary, the sensitivity constant of system (1.1) in V can be any positive number. The proof is complete.

The following example is given to illustrate Theorem 3.1.

Example 3.1. Let I be an unbounded interval in $\mathbb{R}$ and $Q := \{a_n\}_{n=0}^{\infty}$ denote the sequence of rational numbers in I. For each $n \geq 0$, set $V = D_n = X := I$, $f_n(x) = a_n$ for any $x \in Q$, and $f_n(x) = x$ for any $x \in X \backslash Q$. Then $P(f_{0,\infty}) = V \backslash Q$ is dense in V. Let U_0 and V_0 be any two nonempty open subsets of V and fix any $x_0 \in U_0 \cap Q$. Since there exists $n_0 \geq 0$ such that $a_{n_0} \in V_0$, thus $f_0^{n_0+1}(x_0) = a_{n_0} \in V_0$ and hence system (1.1)

is topologically transitive in V. By Theorem 3.1, system (1.1) is sensitive in V and its sensitivity constant can be an arbitrarily positive number.

Now, we consider the other case that V is bounded. The following lemma is needed.

Lemma 3.1. *Let V be an infinite subset of D_0 and $A \subset V$ be dense in V. If system* (1.1) *is not sensitive in V, then for any given $\delta > 0$, there exist $y_0 \in A$ and $\varepsilon_0 \in (0, \delta)$ such that*

$$f_0^n(B_{\varepsilon_0}(y_0) \cap V) \subset B_\delta(f_0^n(y_0)),\ n \geq 0. \tag{3.4}$$

Proof. Fix any $\delta > 0$. Since system (1.1) is not sensitive in V, there exist $x_0 \in V$ and $\varepsilon_0 \in (0, \delta)$ such that $d(f_0^n(y), f_0^n(x_0)) < \delta/2$ for any $y \in B_{2\varepsilon_0}(x_0) \cap V$ and each $n > 0$. Since A is dense in V, there exists $y_0 \in A$ such that $y_0 \in B_{\varepsilon_0}(x_0)$. Then for any $y \in B_{\varepsilon_0}(y_0) \cap V$, $d(y, x_0) \leq d(y, y_0) + d(y_0, x_0) < 2\varepsilon_0$ and thus $d(f_0^n(y), f_0^n(x_0)) < \delta/2$ for each $n > 0$. Hence, $d(f_0^n(y), f_0^n(y_0)) \leq d(f_0^n(y), f_0^n(x_0)) + d(f_0^n(x_0), f_0^n(y_0)) < \delta/2 + \delta/2 = \delta$ for any $y \in B_{\varepsilon_0}(y_0) \cap V$ and each $n > 0$. This, together with the fact that $0 < \varepsilon_0 < \delta$, implies that (3.4) holds. The proof is complete.

Theorem 3.2. *Let V be a bounded and infinite subset of D_0. Assume that $\{f_n\}_{n=0}^\infty$ is equi-continuous in $\{D_n\}_{n=0}^\infty$ and system* (1.1) *has a fixed point in V. If system* (1.1) *is topologically transitive in V and $P(f_{0,\infty}) \cap V$ is dense in V, then it is sensitive in V.*

Proof. Set $\delta = d(V)/7$, where $d(V) := \sup_{x,y\in V} d(x, y) > 0$. Suppose that system (1.1) is not sensitive in V. Then by Lemma 3.1 and the assumption that $P(f_{0,\infty}) \cap V$ is dense in V, there exist $p_0 \in P(f_{0,\infty}) \cap V$ with prime period n_0 and $0 < \varepsilon_0 < \delta$ such that

$$f_0^n(B_{\varepsilon_0}(p_0) \cap V) \subset B_\delta(f_0^n(p_0)),\ n \geq 0. \tag{3.5}$$

Note that there are no isolated points in V by Lemma 2.1. Since system (1.1) is topologically transitive in V and p_0 is a n_0-periodic point in V, by (3.5) one gets that

$$V \subset \overline{\bigcup_{n=0}^{\infty} f_0^n(B_{\varepsilon_0}(p_0) \cap V)} \subset \overline{\bigcup_{n=0}^{\infty} B_\delta(f_0^n(p_0))} = \overline{\bigcup_{n=k_0}^{k_0+n_0-1} B_\delta(f_0^n(p_0))} \tag{3.6}$$

for any fixed $k_0 \geq 0$. Since $\{f_n\}_{n=0}^\infty$ is equi-continuous in $\{D_n\}_{n=0}^\infty$, it can be easily verified that there exists $0 < \varepsilon_1 < \delta$ such that for any $n \geq 0$ and any $x, y \in D_n$ with $d(x, y) < \varepsilon_1$,

$$d(f_n^i(x), f_n^i(y)) < \delta,\ 0 \leq i \leq n_0 - 1. \tag{3.7}$$

Let $q_0 \in V$ be the fixed point of system (1.1). Then $f_n^i(q_0) = q_0$, and thus by (3.7),

$$f_n^i(B_{\varepsilon_1}(q_0) \cap D_n) \subset B_\delta(q_0),\ n \geq 0,\ 0 \leq i \leq n_0 - 1. \tag{3.8}$$

Since system (1.1) is topologically transitive in V, there exists an integer $m_0 > 0$ such that $f_0^{m_0}(B_{\varepsilon_0}(p_0) \cap V) \cap (B_{\varepsilon_1}(q_0) \cap V) \neq \emptyset$. Hence, there exists $y_0 \in B_{\varepsilon_0}(p_0) \cap V$ such that

$$f_0^{m_0}(y_0) \in B_{\varepsilon_1}(q_0) \cap V \cap D_{m_0}. \tag{3.9}$$

By (3.8) and (3.9), one gets that

$$f_0^{m_0+i}(y_0) = f_{m_0}^i \circ f_0^{m_0}(y_0) \in f_{m_0}^i(B_{\varepsilon_1}(q_0) \cap D_{m_0}) \subset B_\delta(q_0),\ 0 \le i \le n_0 - 1. \tag{3.10}$$

Since $y_0 \in B_{\varepsilon_0}(p_0) \cap V$, by (3.5) one has that

$$f_0^{m_0+i}(y_0) \in f_0^{m_0+i}(B_{\varepsilon_0}(p_0) \cap V) \subset B_\delta(f_0^{m_0+i}(p_0)),\ 0 \le i \le n_0 - 1. \tag{3.11}$$

By (3.10) and (3.11), one gets that for any given $0 \le i \le n_0-1$ and any $z_i \in B_\delta(f_0^{m_0+i}(p_0))$,

$$d(z_i, q_0) \le d(z_i, f_0^{m_0+i}(p_0)) + d(f_0^{m_0+i}(p_0), f_0^{m_0+i}(y_0)) + d(f_0^{m_0+i}(y_0), q_0) < 3\delta.$$

This implies that

$$\bigcup_{i=0}^{n_0-1} B_\delta(f_0^{m_0+i}(p_0)) \subset B_{3\delta}(q_0),$$

and thus by (3.6) one gets that

$$V \subset \overline{\bigcup_{n=m_0}^{m_0+n_0-1} B_\delta(f_0^n(p_0))} \subset \overline{B}_{3\delta}(q_0).$$

Hence, $d(V) \le 6\delta$, which contradicts the fact that $d(V) = 7\delta$. The proof is complete.

The following example is given to illustrate Theorem 3.2.

Example 3.2. Let $X = D_n = [0, 1]$, $f_{2n}(x) = x$, and

$$f_{2n+1}(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2], \\ 2(1-x) & \text{if } x \in (1/2, 1], \end{cases}$$

be the tent map for each $n \ge 0$. Set $V = D_0$. Since $f_{2n+1} \circ f_{2n}$ is also the tent map in $[0, 1]$, it is easy to check that system (1.1) satisfies all the conditions in Theorem 3.2. Thus, it is sensitive in V.

For any given positive integer N, denote

$$P_N(f_{0,\infty}) := \{p : p \in P(f_{0,\infty}) \text{ with prime period no more than } N\}.$$

It is evident that the condition that $P_N(f_{0,\infty}) \cap V$ is dense in V for some $N > 0$ is stronger than condition (2) in Definition 2.4.

Theorem 3.3. *Let V be a bounded and infinite subset of D_0. If system* (1.1) *is topologically transitive in V and $P_N(f_{0,\infty}) \cap V$ is dense in V for some $N > 0$, then it is sensitive in V.*

Proof. Since V is infinite, there exist $N+1$ different points $x_i \in V, 0 \le i \le N$, and $\delta > 0$ such that $B_{3\delta}(x_i), 0 \le i \le N$, are mutually disjointed. Suppose that system (1.1) is not sensitive in V. Then with a similar argument to the proof of (3.6), one gets that there exists $p_1 \in P_N(f_{0,\infty}) \cap V$ with prime period $n_1 \le N$ such that

$$V \subset \bigcup_{n=0}^{n_1-1} \overline{B}_\delta(f_0^n(p_1)). \tag{3.12}$$

By noting that $B_{3\delta}(x_i), 0 \leq i \leq N$, are mutually disjointed, it is easy to verify that for each $0 \leq j \leq n_1 - 1$, $\overline{B}_\delta(f_0^j(p_1))$ contains at most one element of $\{x_i\}_{i=0}^N$. Thus there exists some i_0 such that $0 \leq i_0 \leq N$ and $x_{i_0} \notin \cup_{n=0}^{n_1-1}\overline{B}_\delta(f_0^n(p_1))$. This contradicts (3.12). Hence, system (1.1) is sensitive in V. The proof is complete.

Example 3.3. Replace the unbounded interval I in $\mathbb{R}$ by a bounded one in Example 3.1. Then system (1.1) in Example 3.1 is sensitive in V by Theorem 3.3.

4. The special case that $\{f_n\}_{n=0}^\infty$ converges to f

In this section, we shall first study relationships between some chaotic behavior of system (1.1) and that of system (1.2) if the sequence of continuous maps $\{f_n\}_{n=0}^\infty$ converges to a continuous map f and condition (2) in Definition 2.4 holds, and then show that transitivity and density of periodic points imply sensitivity for system (1.1) in this special case. In addition, if f (or f_0) is surjective, then $f_n = f$ for each $n \geq 0$.

We first consider the relationship between periodic points of system (1.1) and those of system (1.2).

Lemma 4.1. *Let V be a totally invariant set of system* (1.1) *and $\{f_n\}_{n=0}^\infty$ converge to f in V. If $x_0 \in V$ is a k_0-periodic point of system* (1.1)*, then $f_0^i(x_0) = f^i(x_0)$ for each $i \geq 0$. Consequently, x_0 is a k_0-periodic point of system* (1.2).

Proof. Let $x_0 \in V$ be a k_0-periodic point of system (1.1) and $i \geq 0$. Then

$$f_0^{i+1}(x_0) = f_0^{nk_0+i+1}(x_0) = f_{nk_0+i}(f_0^{nk_0+i}(x_0)) = f_{nk_0+i}(f_0^i(x_0)),\ n \geq 0.$$

Since V is a totally invariant set of system (1.1) and $f_{nk_0+i} \to f$ in V as $n \to \infty$, $f_0^{i+1}(x_0) = f(f_0^i(x_0))$. By induction, one gets that $f_0^i(x_0) = f^i(x_0)$ for each $i \geq 0$. This completes the proof.

Now, we study the relationship between some chaotic behavior of system (1.1) and that of system (1.2).

Proposition 4.1. *Let V be a totally invariant set of system* (1.1) *and $\{f_n\}_{n=0}^\infty$ converge to a map f in V. If f_n for all $n \geq 0$ and f are continuous in V, and $P(f_{0,\infty}) \cap V$ is dense in V, then*

(i) *$f_0^n = f^n$ in V for each $n \geq 1$;*

(ii) *topological transitivity of system* (1.1) *in V is equivalent to that of system* (1.2) *in V;*

(iii) *sensitivity of system* (1.1) *in* V *is equivalent to that of system* (1.2) *in* V.

Proof. First, we show that (i) holds. Fix any $x \in V$. Since $P(f_{0,\infty}) \cap V$ is dense in V, there exists a sequence $\{p_m\}_{m=0}^{\infty} \subset P(f_{0,\infty}) \cap V$ such that $p_m \to x$ as $m \to \infty$. By Lemma 4.1, we have that

$$f_0^n(p_m) = f^n(p_m),\ m \geq 0,\ n \geq 1. \tag{4.1}$$

Letting $m \to \infty$ in (4.1) for any given $n \geq 1$, one has that $f_0^n(x) = f^n(x)$ by the continuity of f_0^n and f^n. Thus assertion (i) holds.

Assertions (ii) and (iii) can be shown by (i) immediately. This completes the proof.

Next, we shall show that the two former conditions imply the last one in Definition 2.4 by Lemma 2.2 and Proposition 4.1 under some assumptions.

Theorem 4.1. *Let* V *be a totally invariant set of system* (1.1) *and infinite. Assume that* $\{f_n\}_{n=0}^{\infty}$ *converges to a map* f *in* V, *and* f_n *for all* $n \geq 0$ *and* f *are continuous in* V. *If system* (1.1) *is topologically transitive in* V *and* $P(f_{0,\infty}) \cap V$ *is dense in* V, *then it is sensitive in* V.

Proof. Since system (1.1) is topologically transitive in V, system (1.2) is also topologically transitive in V by (ii) of Proposition 4.1. Because $P(f_{0,\infty}) \cap V$ is dense in V, it follows from Lemma 4.1 that $P(f) \cap V$ is dense in V. By Lemma 2.2, system (1.2) is sensitive in V. Therefore, system (1.1) is sensitive in V by (iii) of Proposition 4.1. The proof is complete.

Remark 4.1. By Lemma 4.1 and Proposition 4.1, Devaney chaos of system (1.1) implies that of system (1.2) under the assumptions of Theorem 4.1. However, the converse is not true. See the following example:

Example 4.1. Let $V = D_n = X := [0,1]$,

$$f_0(x) = \begin{cases} 2x & \text{if } x \in [0,1/3], \\ 2/3 & \text{if } x \in (1/3,2/3), \\ x & \text{if } x \in [2/3,1], \end{cases} \qquad f = f_n(x) = \begin{cases} 2x & \text{if } x \in [0,1/2], \\ -2x+2 & \text{if } x \in (1/2,1], \end{cases}$$

where $n \geq 1$. Then $\{f_n\}_{n=0}^{\infty}$ converges to f in V. Let $U_0 = (1/3,2/3)$ and $V_0 = (0,1/3)$. It is clear that $f_0^n(U_0) = \{2/3\}$ and hence $f_0^n(U_0) \cap V_0 = \emptyset$ for each $n > 0$. This implies that system (1.1) is not chaotic in the sense of Devaney. However, f is the tent map in $[0,1]$ and hence system (1.2) is chaotic in the sense of Devaney.

The following result shows that $f_n = f$ in V for each $n \geq 0$ if all the conditions in Proposition 4.1 are satisfied and in addition f_0 is surjective in V.

Proposition 4.2. *Let* V *be a totally invariant set of system* (1.1) *and* f_n *be continuous in* V *for each* $n \geq 0$. *Assume that* $P(f_{0,\infty}) \cap V$ *is dense in* V *and* f_0 *is surjective in* V.

Then $\{f_n\}_{n=0}^{\infty}$ converges to a continuous map f in V if and only if $f_n = f$ in V for each $n \geq 0$.

Proof. It suffices to show the necessity. By (i) of Proposition 4.1, $f_0 = f$ in V. Thus f is surjective in V. Suppose that for $k \geq 0$, $f_j = f$ in V for each $0 \leq j \leq k$. Then $f_j, 0 \leq j \leq k$, are surjective in V. For any $y \in V$, there exist $x_j \in V$, $0 \leq j \leq k$, such that $y = f_k(x_k) = f(x_k)$, $x_{i+1} = f_i(x_i) = f(x_i)$, $0 \leq i \leq k-1$. It follows from (i) of Proposition 4.1 that $f_{k+1}(y) = f_0^{k+2}(x_0) = f^{k+2}(x_0) = f(y)$, which implies that $f_{k+1} = f$ in V. Hence, the necessity holds by induction. This completes the proof.

The following result is a direct consequence of Propositions 4.1 and 4.2.

Corollary 4.1. *Let all the conditions in Proposition* 4.1 *hold. Assume that f is surjective in V. Then $f_n = f$ for each $n \geq 0$.*

In the case that the metric space is compact, we have the following result:

Proposition 4.3. *Let (X, d) be a compact metric space and $f_n : X \to X$ be continuous for each $n \geq 0$. Assume that system* (1.1) *is topologically transitive and $P(f_{0,\infty})$ is dense in X. Then the following statements are equivalent:*

(i) *$f_n = f$ for each $n \in \mathbb{N}$;*

(ii) *$\{f_n\}_{n=0}^{\infty}$ uniformly converges to a map f in X;*

(iii) *$\{f_n\}_{n=0}^{\infty}$ converges to a continuous map f in X.*

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (i): By Corollary 4.1, it suffices to show that f is surjective in X. Otherwise, $f(X) \subsetneqq X$. Since f is continuous in X and X is compact, $f(X)$ is compact and thus closed in X. Hence, $X \backslash f(X)$ is a nonempty open subset in X. By (i) of Proposition 4.1, $f_0^n(U_0) = f^n(U_0) \subset f(X)$ for any fixed nonempty open subset U_0 and each $n > 0$. Thus, $f_0^n(U_0) \cap (X \backslash f(X)) = \emptyset$ for each $n > 0$. This contradicts the assumption that system (1.1) is topologically transitive in X. Therefore, f is surjective in X. The proof is complete.

References

[1] E. Akin, Recurrence in Topological Dynamics: Furstenberg Families and Ellis Actions, Plenum Press, New York, 1997.

[2] Z. AlSharawi, J. Angelos, S. Elaydi, L. Rakesh, An extension of Sharkovskys theorem to periodic difference equations, J. Math. Anal. Appl. 316 (2006) 128–141.

[3] J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey, On Devaney's definition of chaos, Amer. Math. Monthly 99 (1992) 332–334.

[4] F. Balibrea, P. Oprocha, Weak mixing and chaos in nonautonomous discrete systems, Appl. Math. Lett. 25 (2012) 1135–1141.

[5] F. Balibrea, J. Smital, M. Štefánková, The tree versions of distributional chaos, Chaos Solitons Fract. 23 (2005) 1581–1583.

[6] J. S. Cánovas, Li-Yorke chaos in a class of nonautonomous discrete systems, J. Differ. Equ. Appl. 17 (2011) 479–486.

[7] R. L. Devaney, An Introduction to Chaotic Dynamical Systems, Addison–Wesley, 1989.

[8] J. Dvořáková, Chaos in nonautonomous discrete dynamical systems, Commun. Nonlinear Sci. Numer. Simulat. 17 (2012) 4649–4652.

[9] H. Furstenberg, Recurrence in Ergodic Theory and Combinatorial Number Theory, Princeton Univ. Press, Princeton, 1981.

[10] E. Glasner, B. Weiss, Sensitive dependence on initial conditions, Nonlinearity 6 (1993) 1067–1075.

[11] Q. Huang, Y. Shi, L. Zhang, Sensitivity of non-autonomous discrete dynamical systems, Appl. Math. Lett. 39 (2015) 31–34.

[12] W. Huang, X. Ye, An explicit scattering, non-weakly mixing example and weak disjointness, Nonlinearity 15 (2002) 849–862.

[13] W. Huang, X. Ye, Devaneys chaos or 2-scattering implies Li-Yorkes chaos, Topol. Appl. 117 (2002) 259–272.

[14] S. Kolyada, L. Snoha, Topological entropy of non-autononous dynamical systems, Random Comp. Dyn. 4 (1996) 205–233.

[15] T. Li, J. Yorke, Period three implies chaos, Amer. Math. Monthly 82 (1975) 985–992.

[16] M. Pourbarat, N. Abbasi, Nonautonomous discrete systems: density of orbits and transitivity, Int. J. Bifurcat. Chaos 25 (2015) 1550065, 6 pages.

[17] C. Robinson, Dynamical systems, Stability, Symbolic dynamics, and Chaos, second ed., CRC press, 1999.

[18] B. Schweizer, J. Smital, Measures of chaos and a spectral decomposition of dynamical systems on the interval, Trans. Amer. Math. Soc. 344 (1994) 737–854.

[19] H. Shao, Y. Shi, H. Zhu, Strong Li-Yorke chaos for time-varying discrete dynamical systems with A-Coupled-Expansion, Int. J. Bifurcat. Chaos 25 (2015) 1550186, 10 pages.

[20] H. Shao, Y. Shi, H. Zhu, Estimations of topological entropy for non-autonomous discrete systems, J. Differ. Equ. Appl., online, DOI: 10.1080/10236198.2015.1107055.

[21] Y. Shi, G. Chen, Chaos of time-varying discrete dynamical systems, J. Differ. Equ. Appl. 15 (2009) 429–449.

[22] Y. Shi, Chaos in nonautonomous discrete dynamical systems approached by their induced systems, Int. J. Bifurcat. Chaos 22 (2012) 1250284, 12 pages.

[23] Y. Shi, L. Zhang, P. Yu, Q. Huang, Chaos in periodic discrete systems, Int. J. Bifurcat. Chaos 25 (2015) 1550010, 21 pages.

[24] C. Tian, G. Chen, Chaos of a sequence of maps in a metric space, Chaos Solitons Fract. 28 (2006) 1067–1075.

[25] S. Wiggins, Introduction of Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag, New York, 1990.

[26] X. Wu, P. Zhu, Chaos in a class of non-autonomous discrete systems, Appl. Math. Lett. 26 (2013) 431–436.